Refinements and Extensions of the Euler Partition Theorem
(lecture notes)

Ivica Martinjak
University of Zagreb
Zagreb, Croatia

Abstract

Firstly we discuss and prove some basic properties of integer partitions as recursion $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$, where $p_k(n)$ is the number of partition $\lambda \vdash n$ into k parts, Bressoud's identity for 2-distinct partition etc. The introduction is completed with the proof of Euler theorem of equality of partition into odd and distinct parts. By the Franklin bijection we prove the *Euler pentagonal number theorem* and some of its refinements. Sylvester's bijection is used to prove one of Fine's refinement of Euler theorem. The notion *rank of a partition* is introduced and Dyson's bijection is illustrated on a concrete example. Finally we present the *lecture hall theorem* as a finite version of Euler's theorem.

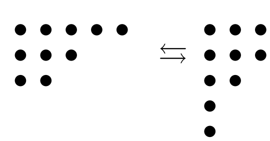
1 Introduction

A partition of the natural number n is the sequence $\lambda=(\lambda_1, \lambda_2, ..., \lambda_l)$, with condition $\lambda_i \geq \lambda_{i+1}$ where $\lambda_i \in \mathbb{N}_0$, $\sum_{i=1}^{l} \lambda_i=|\lambda|=n$, denoted $\lambda \vdash n$. The numbers λ_i are called the parts of a partition, the number of parts $l(\lambda)=l$ we call the length of a partition. Let $l_o(\lambda)$ denots the number of odd parts of a partition. Let $e(\lambda)$ and $a(\lambda)$ be any even part and the greatest part amoung parts of a partition $\lambda \vdash n$, respectively.

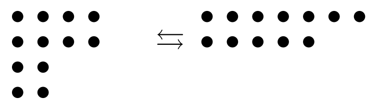
Let the set of partitions $\lambda \vdash n$ is denoted by $\mathcal{P}_n$. Let $\mathcal{D}_n$ be the set of partitions with mutually distinct parts and $\mathcal{O}_n$ the set of partitions with all parts odd.

We are going to consider some basic facts on partitions.

Proposition 1 *(a) The number of partition $\lambda \vdash n$ in l parts equals the number of partitions $\mu \vdash n$ with l the greatest part, $p(n|l(\lambda) = l) = p(n|a(\mu) = l)$.*



(b) $p(n|self\text{-}conjugate) = p(n|parts\ distinct\ and\ odd)$



(c) $p_l(n) = p_{l-1}(n-1) + p_l(n-l)$

(d) $p(n) = p_n(2n)$

Proof Proofs of statemets (a) and (b) are evident from the figures above.

(c) If the partition $\lambda \vdash n$ has the number 1 as a part, than subtructing it gives $p_{l-1}(n-1)$ partitions. If this is not the case, then we subtract 1 from every part, obtaining $p_l(n-l)$ partitions.

(d) The statement follows from the previous one: $p_n(2n) = p_{2n-1}(2n-1) + p_n(n) = p_{n-2}(2n-2) + p_{n-1}(n) + p_n(n) = p_1(n) + ... + p_{n-1}(n) + p_n(n) = p(n)$.

As an illustration of statement (d) we can say that $p(4) = p_4(8)$, i.e. there is 5 partition of 8 having 4 parts:

$$(5,1,1,1), (4,2,1,1), (3,3,1,1), (3,2,2,1), (2,2,2,2).$$

2 Euler's pentagonal number theorem

As we now, the j-th triangular number is $\frac{j(j+1)}{2}$. Having in mind that the j-th pentagonal number is built from $(j-1)$-th triangular number and j^2 (see the graphs bellow), for the j-th pentagonal number it holds

$$jj + (j-1)(j-1+1)/2 = j^2 + (j^2-1)/2 = j(3j-1)/2$$

Considering 1-distinct partitions $\lambda \vdash n$, one can noticed that the number of partitions with even parts equals the partitions with odd parts; with exception of $n = j((3j-1)/2$ being pentagonal number or its j-*th* succesor.

Theorem 1 *(Euler) Let $\mathcal{D}_n^o$ and $\mathcal{D}_n^d$ be sets of partitions into odd distinct and even distinct parts, respectively. Then it holds*

$$|\mathcal{D}_n^o| = |\mathcal{D}_n^d| + E(n), \ \ E(n) = \begin{cases} (-1)^j, & n{=}j(3j \pm 1)/2 \\ 0, & otherwise \end{cases}.$$

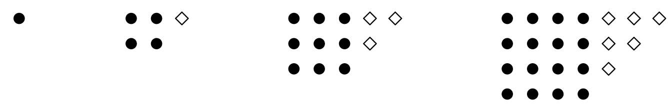


Figure 1: The first three pentagonal numbers. The j-th pentagonal number is consisted by the square of j and the (j-1)-th triangular number.

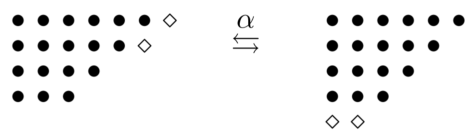


Figure 2: The map α is a bijection except for the pentagonal number $n = j(3j+1)/2$ and its j-th successor.

Proof (Franklin bijection, 1881.)

The function $\alpha : \mathcal{D}_n \leftarrow \mathcal{D}_n$ keeps parts distinct and changes the parity of $l(\lambda)$, but not for all n. Two exceptions are $j(3j-1)/2$ and $j(3j+1)/2$. In the latter, if the last row is cut and paste to diagonal - the parity stays unchanged; if we cut the diagonal and paste it on the buttom - we disturb the structure of Ferrers graph. An analogy holds for the former.

Note that the bijection α has one more property: it changes the parity of $a(\lambda)$. This fact is expressed by the next theorem.

Theorem 2 *(Fine) The number of partitions into distinct parts having the greatest part even equals the number of partitions into distinct parts having the greatest part odd.*

$$|\mathcal{D}_n : a(\lambda) \equiv 0(mod 2)| + |\mathcal{D}_n : a(\lambda) \equiv 1(mod 2)| + \begin{cases} 1, & n = j(3j+1)/2 \\ -1, & n = j(3j-1)/2 \\ 0, & otherwise \end{cases}$$

As the Example 1 presents, there are 8 partitions of 9 with distinct parts, four of them with even length and four of them with odd parts. Furthermore, according to the Fine's theorem 4 out of these paritions have the greatest part even $(6, 4, 8$ and $6)$ and there are the same number of odd greatest parts $(9, 5, 7$ and $5)$.

Example 1 *n=9*

$\mathcal{D}_9^e$	$\mathcal{D}_9^o$
$(8,1)$	(9)
$(7,2)$	$(6,2,1)$
$(6,3)$	$(5,3,1)$
$(5,4)$	$(4,3,2)$

3 Sylvester's bijection

Let $\lambda = (7,5,5,3) \in \mathcal{O}_n$. We define the function $\phi : \mathcal{O}_n \to \mathcal{D}_n$ as the next figure illustrates. Since we always can represent an odd number as $1+2q$, $q \in \mathbb{Z}$, ϕ is a bijection. According to the Figure 4, it holds

$$a(\mu) = \frac{a(\lambda)-1}{2} + l(\lambda) \;\;\Rightarrow\;\; a(\lambda) + 2l(\lambda) = 2a(\mu) + 1$$

This proves the next theorem.

Theorem 3 *(Fine) The number of partitions $\mu \vdash n$, $a(\mu) = m$ into distinct parts equals the number of partitions $\lambda \vdash n$ into odd parts with $a(\lambda)+2l(\lambda) = 2m+1, m > 0$.*

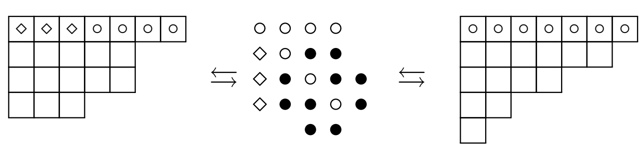


Figure 3: Sylvester bijection for $\lambda = (7,5,5,3)$, $\mu = (7,6,4,2,1)$.

As an example, let consider the case $n = 11,\ m = 8$. The set $\mathcal{O}_{11}$: $a(\lambda) + 2l(\lambda) = 17$ counts two elements: $(7,1,1,1,1),(3,3,1,1,1,1,1)$. The same is case within the set $\mathcal{D}_{11}$: $a(\mu) = 8$, these partitions are $(8,3)$ and $(8,2,1)$. The whole list of $\mathcal{D}_{11}$ and $\mathcal{O}_{11}$ see in the Example 2.

4 Rank of a partition and Fine's refinement of the Euler theorem

The *rank* $r(\lambda)$ of a partition $\lambda \vdash n$ is defined as the difference between the largest part $a(\lambda)$ and the number of parts $l(\lambda)$ (F.Dyson), $r(\lambda) = a(\lambda) - l(\lambda)$.

The set of partitions of n with rank r we denote $\mathcal{P}_{n,r}$. The set of partitions of n with rank at most r we denote $\mathcal{H}_{n,r}$ while the set of partitions of n with rank at least r is denoted by $\mathcal{G}_{n,r}$. Obviously, it holds

$$\begin{aligned} p(n,r) &= h(n,r) - h(n,r-1) \\ g(n,r) &= h(n,-r) \end{aligned}$$

where $p(n,r), h(n,r), g(n,r)$ are cardinalities of the previously defined sets, respectively.

Theorem 4 *(Fine)* $h(n, 1+r) = h(n+r, 1-r)$

Proof (Dyson's bijection) Let $\psi_r : \mathcal{H}_{n,r+1} \to \mathcal{G}_{n+r,r-1}$, where ψ_r cuts the first column of the Young diagram of a starting partition λ and paste it on the top of the rest of the partition, togather with r squares in addition (see the figure below).

Now we have to prove that the resulting partition $\mu \in \mathcal{G}_{n+r,r-1}$, i.e. that it holds *i)* $|\mu| = n + r$ and *ii)* $r(n) \geq r - 1$ where the latter fact is obvious. Having $r(\mu) = a(\mu) - l(\mu) = l(\lambda) + r - l(\mu)$ and $l(\mu) =$ (the length of second column of λ) $+ 1$, the condition *ii)* follows immidiately.

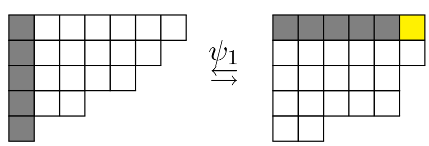


Figure 4: Dyson bijection, $\lambda = (7, 6, 6, 3, 1) \in \mathcal{H}_{23,r+1}$, $\mu = (6, 6, 5, 5, 2) \in \mathcal{G}_{23+r,r-1}$.

Using this bijection iteratively one can prove the next remarkable refinement of the Euler theorem. Namely, not only $\mathcal{O}_n = \mathcal{D}_n$ but there are subsets of both type of partitions, with respect to a rank, having the same cardinality for a given rank.

Theorem 5 *(Fine) The number of partitions* $\lambda \vdash n$ *into distinct parts,* $r(\lambda) = r$ *or* $r(\lambda) = 2r + 1$ *equals the number of partitions* $\mu \vdash n$ *into odd parts having the greatest part* $2r + 1$*,*

$$|\mathcal{O}_{n,a(\mu)=2r+1}| = |\mathcal{D}_{n,2r} + |\mathcal{D}_{n,2r+1}|.$$

So, instead of a proof we will use the Dyson bijection iteratively in the next example in order to illustrate the theorem. In the first iteration draw the 1-row diagram representing the smallest part of the starting partition. Furthermore, apply the ψ_r where r equals the next smallest part in the partition.

Example 2

r	$\mathcal{D}_{11}$	$\mathcal{O}_{11}$
5	(11)	(11)
4	$(10,1)$	$(9,11)$
3	$(9,2),(8,3)$	$(7,3,1),(7,1,1,1)$
2	$(8,2,1),(7,4),(7,3,1),(6,5)$	$(5,5,1),(5,3,3),(5,3,1,1,1),(5,1^6)$
1	$(6,4,1),(6,3,2),(5,4,2)$	$(3,3,3,1,1),(3,3,1^5),(3,1^8)$
0	$(5,3,2,1)$	(1^{11})

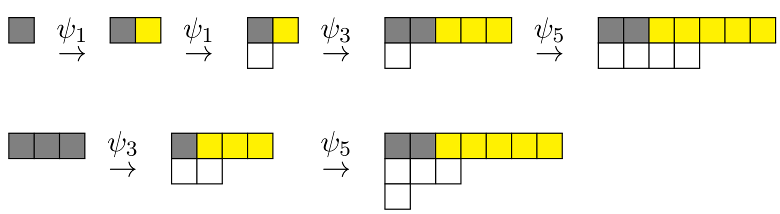


Figure 5: Iterative Dyson bijection, from $\lambda = (5,3,1,1,1)$ to $\mu = (7,4)$ and $\lambda' = (5,3,3)$ to $\mu' = (7,3,1)$, which are two out of four possibilities for n=11, $r(\lambda)$=2.

5 Lecture hall partitions

Definition 1 *Let* $\mathcal{L}_N = \{\lambda_1 + \lambda_2 + ... + \lambda_N : 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq ... \leq \frac{\lambda_N}{N}\}$. *The set* $\mathcal{L}_N$ *we call the lecture hall partitions of length* N.

The name of the set $\mathcal{L}_N$ suggests that one can imagine λ_i as the heights of seats in a lecture hall. For example, $(1,2,4)$ is a lecture hall partition of 3 since $0 \leq \frac{1}{1} \leq \frac{2}{2} \leq \frac{4}{3}$.

Theorem 6 *(Bousquet-Mélou, Eriksson, 1995)*

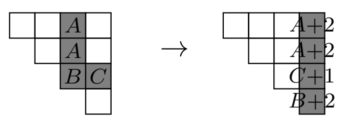
$$p(n|\textit{lecture hall partition of } N) = p(n|\textit{even parts} < 2N)$$

The theorem can be understand as the finite version of the Euler's theorem. Namely, when $N \to \infty$ on the right hand side of the equation we have $\mathcal{O}_n$. On the l.h.s. we have $\frac{\lambda_{N-k}}{N-k} \leq \frac{\lambda_{N-k+1}}{N-k+1}$ which reduces to $\lambda_{N-k} < \lambda_{N-k+1}$ for $N \to \infty$, giving the set $\mathcal{D}_n$.

Instead of the full proof of this theorem, that uses Coxeter groups, we will present a nice bijection between partitions having distinct parts $\leq N$ and lecture hall partitions. Let $N \in \mathbb{N}$, $\mu = \mu_1 + \mu_2 + ... + \mu_k$, $\mu_1 \leq \mu_2 \leq ... \leq \mu_k$.

The initial step. Form the partition $\mu_0 = \mu_1 + \mu_2 + ... + \mu_m$, $\mu_m \leq N$, $\mu_{m+1} > N$. Form the triangular matrix $N \times N$, with the first μ_i entries 1 counting from the bottom, in i-th column.

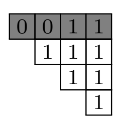
i-th step. Determine the number j, $\mu_j = \mu[m+i] = N + j$. Cut the j-th column and the j-th row in the current matrix and paste it according to the next scheme.



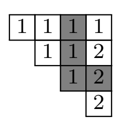
The last step. The sum per columns gives the lecture hall partition.

Example 3 $\mu = 1 + 3 + 4 + 5 + 7 + 7$, $N = 4$

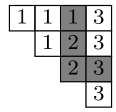
Initial step. $\mu_0 = 1 + 3 + 4$



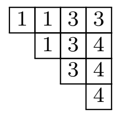
1st step. $\mu_1 = 5 = 4 + 1$



2st step. $\mu_2 = 7 = 4 + 3$



3st step. $\mu_3 = 7 = 4 + 3$



Finally, the sum per columns gives the lecture hall partition $(1, 2, 9, 15)$*.*

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